Errors in a Magic-Tee Phase Changer*

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Summary—**This paper recalls the basic properties of a magic-tee and how it can be used as a linear phase changer. An analysis of the symmetrical magic-tee phase changer is made, which shows that nonlinearities of the phase shift and amplitude modulation are second and higher order effects caused by small mismatches of the structure. Also, some qualitative comments are made on the errors of an asymmetrical phase changer. Measurements on a phase changer assembled from ordinary laboratory equipment show that the phase shift is linear to better than 1°.**

Introduction

THE PRINCIPLE of operation of the ideal magic-tee phase changer[1] is readily understood from a consideration of the properties of a magic-tee. Many microwave structures have the properties of a magic-tee[2,3] but for the sake of simplicity, the most familiar device, *i.e.*, the side outlet "*E*" and "*H*" plane tee is used to visualize the principle of operation. The side outlet tee shown in Fig. 1, which is assumed to be symmetrical with respect to the plane which contains the axis of arms 3 and 4, is called a magic-tee when the structure is completely matched, *i.e.*, when a matched load is seen looking into any one of the four arms if the other are terminated in reflectionless loads. For this case, the total power a generator feeds into any one arm couples in equal amounts into the two adjacent arms and not into the opposite arm. If a generator is connected to arm 4, the coupled waves into arms 1 and 2 have the same phase in planes equidistant from the symmetry plane, but are π radians out of phase if the generator feeds arm 3. Now, if two variable short circuits are inserted into arms 1 and 2 at a distance L_1, and $L_2 = L_1 + (\lambda g/4)$ respectively from the symmetry plane, all the power going into arm 4 will appear at the output of arm 3. The wave launched at input 4 splits into two equal and in phase waves in arms 1 and 2. Because arm 2 is $\lambda g/4$ longer than arm 1, these two waves are reflected back in the symmetry plane with a phase difference of π radians and will cancel each other in arm 4 but reinforce each other in arm 3. If the lengths of arms 1 and 2 are both increased by ΔL, the total path length of the structure has changed by $2\Delta L$, and the phase of the output wave in arm 3 has changed by $(2\Delta L/\lambda g)2\pi$.

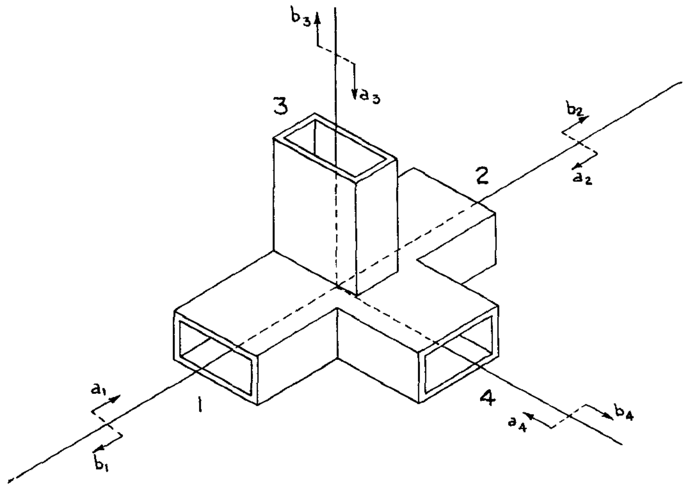

Fig. 1—Side outlet "*E*" and "*H*" plane arm tee.

It should be added that the variable short circuits can also be inserted in arms 3 and 4 instead of arms 1 and 2 and will give a similar relative phase change of the output in arm 2 when arm 1 is connected to a generator.

Ten independent equations are available to fix the relationship between the 16 parameters required to describe the behavior of an unmatched and asymmetrical hybrid tee. The relations between the various parameters are not linear, and solutions for the dependent parameters in terms of a suitably chosen set of six independent parameters are practically untractable. On the other hand, for an unmatched symmetrical tee, only nine parameters are required, and three of them can be chosen independently.

The resulting simplification is such that an analysis of the errors in the linearity of the phase change caused by reflections is possible. Thus the main part of this paper is concerned with the analysis of the errors in a symmetrical hybrid tee phase changer. Some qualitative remarks on the effects of small asymmetries are made at the end of the paper. Experimental results on a typical phase changer assembled from commercial components are also included.

Scattering Matrix of Side-Outlet Tee[4]

The reference planes in each arm (see Fig. 1) are chosen far enough from the junction to be out of the diffraction field. It is assumed that only the fundamental H_{10} mode can be propagated in the waveguides and that the guides have the same uniform cross section and negligible losses.

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[1] G. Saxton and C. W. Miller, "Magic-tee waveguide junction," *Wireless Eng.*, vol. 25, pp. 138–147; May, 1948.

[2] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits, M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, ch. 12; 1948.

[3] W. K. Kahn, "*E*-plane forked hybrid-T junction," IRE Trans., vol. MTT-3, pp. 52–58; December, 1955.

[4] C. G. Montgomery, "Technique of Microwave Measurements," McGraw-Hill Book Co., Inc., New York, N. Y., sec. 9.2; 1947.

The amplitudes of emergent and incident waves are related as follows:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}. \quad (1)$$

a_n and b_n are so normalized that $|a_n|^2$ and $|b_n|^2$ are respectively proportional to the incident and emergent power at terminal n. Since the structure is assumed lossless, its scattering matrix has the properties of a unitary matrix, *i.e.*,

$$\sum_{j=1}^{4} S_{ij}S_{sj}^* = \begin{matrix} 1 & \text{for } i = s \\ 0 & \text{for } i \neq s. \end{matrix} \quad (2)$$

The application of these conditions to the scattering matrix (1) together with the reciprocity conditions

$$S_{ij} = S_{ji}, \qquad i, j, = 1, 2, 3, 4 \quad (3)$$

gives 10 independent equations. But in general, $S_{mn} = U_{mn} + jV_{mn}$, and hence there are 20 unknown parameters. Without loss in generality, the four reference planes can always be chosen so that the diagonal elements S_{mm} are pure real numbers.[5] This choice of reference planes reduces the number of unknown parameters to 16. This implies that at least six unknowns must be measured to evaluate all the others. The above properties of the scattering matrix will be used in the analysis of the magic-tee phase changer.

Analysis of Magic-Tee Phase Changer

Consider a tee with variable plungers inserted in arms 1 and 2 at θ_1 and θ_2 electrical degrees from their respective reference planes. If a generator is connected to arm 4 with arm 3 terminated in a matched load, it follows that

$$a_1 = -b_1 e^{-2j\theta_1}$$
$$a_2 = -b_2 e^{-2j\theta_2}$$
$$a_3 = 0 \quad (4)$$

and the expression for b_3 as a function of a_4 is

$$b_3 = a_4\Big[S_{34} - \frac{1}{\Delta}\{S_{13}S_{14}e^{-2j\theta_1} + S_{23}S_{24}e^{-2j\theta_2} + (-S_{12}S_{13}S_{24} - S_{12}S_{14}S_{23} + S_{13}S_{14}S_{22} + S_{23}S_{24}S_{11})e^{-2j(\theta_1+\theta_2)}\}\Big] \quad (5)$$

[5] Montgomery, Dicke, and Purcell, *op. cit.*, p. 149.

where

$$\Delta = (1 + S_{11}e^{-2j\theta_1})(1 + S_{22}e^{-2j\theta_2}) - S_{12}^2 e^{-2j(\theta_1+\theta_2)}. \quad (6)$$

Any actual tee should have very little asymmetry in order to have a simple calibration law when used as a phase changer. Assuming perfect symmetry makes the analysis possible and yet gives a good theoretical insight to the actual performance of a good magic-tee phase changer.

From symmetry considerations it follows that

$$S_{23} = -S_{13} \qquad S_{22} = S_{11}$$
$$S_{24} = S_{14} \qquad S_{34} = 0. \quad (7)$$

S_{34} must be equated to zero since, for the H_{10} mode of propagation, the electric field distribution in the cross sections of arms 3 and 4 are asymmetrical and symmetrical respectively and hence there is no coupling between these arms. Substitution of (7) in (5) and (6) gives

$$b_3 = \frac{-a_4 S_{13}S_{14}(e^{-2j\theta_1} - e^{-2j\theta_2})}{1 + S_{11}(e^{-2j\theta_1} + e^{-2j\theta_2}) + (S_{11}^2 - S_{12}^2)e^{-2j(\theta_1+\theta_2)}}. \quad (8)$$

The phase and amplitude of b_3/a_4 can be evaluated from (8), but simpler expressions follow without loss in generality if the diagonal elements of the scattering matrix are made pure real numbers. The Appendix gives the derivation of $|S_{12}|$, $|S_{13}|$, and $|S_{14}|$ as functions of U_{11}, U_{33}, and U_{44}, *i.e.*,

$$|S_{13}|^2 = \frac{1 - U_{33}^2}{2}; \qquad |S_{14}|^2 = \frac{1 - U_{44}^2}{2}$$

$$U_{12} = \frac{U_{44}^2 - U_{33}^2}{4U_{11}};$$

$$V_{12}^2 = \frac{U_{33}^2 + U_{44}^2}{2} - U_{11}^2 - U_{12}^2.$$

Substituting these values in (8) the phase of b_3/a_4 is then expressible as

$$\measuredangle \frac{b_3}{a_4} = \phi_0 - \left(\theta_1 + \theta_2 - \frac{\pi}{2}\right) - \psi \quad (9)$$

where

$$\psi = \tan^{-1} - \frac{[U_{11}(\sin 2\theta_1 + \sin 2\theta_2) + \alpha \sin 2(\theta_1 + \theta_2) + \beta \cos 2(\theta_1 + \theta_2)]}{[1 + U_{11}(\cos 2\theta_1 + \cos 2\theta_2) + \alpha \cos 2(\theta_1 + \theta_2) - \beta \sin 2(\theta_1 + \theta_2)]}$$

with

$$\alpha = U_{11}^2 - U_{12}^2 + V_{12}^2$$

and

$$\beta = 2U_{12}V_{12}.$$

The constant phase angle ϕ_0 is the phase of $S_{13}S_{14}$. The modulus of b_3/a_4 is given by

$$\left|\frac{b_3}{a_4}\right| = \frac{2|S_{13}|\,|S_{14}|\,|\sin(\theta_1-\theta_2)|}{\{(1-\alpha)^2+\beta^2+4U_{11}\cos(\theta_1-\theta_2)\cos(\theta_1+\theta_2) + 4[[\cos(\theta_1+\theta_2)+U_{11}\cos(\theta_1-\theta_2)][\alpha\cos(\theta_1+\theta_2)-\beta\sin(\theta_1+\theta_2)]]\}^{1/2}}. \quad (10)$$

Substitution of $\theta_2=\theta_1+(\pi/2)+t$, ($t$ is a small angular error which may arise in the initial setting of the plungers in arms 1 and 2) in (9) and (10) gives

$$\measuredangle\frac{b_3}{a_4} = \phi_0 - (2\theta_1+t) - \psi_1 = \phi \quad (11)$$

where

$$\psi_1 = \tan^{-1}\frac{-U_{11}[\sin 2\theta_1 - \sin 2(\theta_1+t)] + \alpha\sin 2(2\theta_1+t) + \beta\cos 2(2\theta+t)}{1+U_{11}[\cos 2\theta_1 - \cos 2(\theta_1+t)] - \alpha\cos 2(2\theta+t) + \beta\sin 2(2\theta_1+t)}$$

$$\left|\frac{b_3}{a_4}\right| = \frac{2|S_{13}|\,|S_{14}|\cos t}{\{(1-\alpha)^2+\beta^2+4U_{11}\sin t\sin(2\theta_1+t)+4[\sin(2\theta_1+t)+U_{11}\sin t][\alpha\sin(2\theta_1+t)+\beta\cos(2\theta_1+t)]\}^{1/2}}. \quad (12$$

The third term, ψ_1, of (11) is a contribution of second order provided t, U_{11}, U_{12}, and V_{12} are small. Moreover, if t is a small error, its variation δt with plunger setting will be of second order. Thus to a second order of approximation, the phase of b_3/a_4 is a linear function of θ_1 since ϕ_0 is constant. The extrema of the third term should be very near and of the order of magnitude of those obtained for the case $t=0$. For $t=0$

$$\psi_1 = \tan^{-1}\frac{\alpha\sin 4\theta_1+\beta\cos 4\theta_1}{1-\alpha\cos 4\theta_1+\beta\sin 4\theta_1} \quad (13)$$

and ψ_1 has extrema for

$$\theta_1 = \frac{1}{4}\cos^{-1}\left\{\alpha \pm \beta\left[\frac{1}{\alpha^2+\beta^2}-1\right]^{1/2}\right\}. \quad (14)$$

To give an order of magnitude of the expected error due to ϕ_1, let $U_{11}=U_{33}=U_{44}=0.1$ (this implies a residual vswr$=1.22$) and ϕ_1 max$\simeq 0.57°$. When $t=0$

$$\left|\frac{b_3}{a_4}\right| = \frac{2|S_{13}|\,|S_{14}|}{\{1+\alpha^2+\beta^2-2\alpha\cos 4\theta_1+2\beta\sin 4\theta_1\}^{1/2}}$$

$$= 2|S_{13}|\,|S_{14}|\left\{1-\frac{1}{2}(\alpha^2+\beta^2)+\sqrt{\alpha^2+\beta^2}\cdot\left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos 4\theta_1 - \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\sin 4\theta_1\right]+\cdots\right\}$$

$$\approx 2|S_{13}|S_{14}|\,[1+\sqrt{\alpha^2+\beta^2}\cos(4\theta_1+\phi_1) \quad (15)$$

where

$$\phi_1 = \tan^{-1}\beta/\alpha.$$

Since α^2 and β^2 are small quantities of fourth order, the square root is of second order and hence the modulation amplitude $\sqrt{\alpha^2+\beta^2}$ of the output is of second order. The output varies at twice the rate of the phase shift of the principal wave. For the particular values of U_{11}, U_{33}, U_{44} used above the modulation of the output is 1 per cent.

Experimental Verification

To use a side outlet tee as a phase changer it is necessary to match it reasonably well (vswr<1.3) and to have the plungers in the "through" arms respectively at θ_1 and θ_1+90 electrical degrees from the symmetry plane.

For narrow-band operation, traveling screw tuners may be used in the "E" and "H" plane arms. A residual vswr$\simeq 1.03$ is readily obtained provided the load terminations in the other three arms have small "phasable" reflection coefficients. Long tapered wood loads which can be slid in the waveguides do very well.

To fix the respective locations of the plungers in the "through" arms, one reflectionless load termination of the through arms is replaced by a plunger and the position of one minimum of the electric field in the input slotted line is measured. Repeating the same experiment for the other arm with another plunger gives the difference in the electrical length of the two through arms. From this result and the measure of the guide wavelength (not the slotted guide wavelength), one of the two plunger arms can be made 90 electrical degrees longer than the other.

Because the structure was not perfectly matched, the locations of the electric field minima in the slotted line were averaged for open and short-circuit conditions at the output by moving the output plunger a quarter of a guide wavelength away from its initial position for every setting of the plungers in the through arms.

The results of the measurements are shown in Table I. The maximum variation of the incremental phase shift

TABLE I

$2L_1$ cm	ϕ	$\Delta\phi$
0.000	0.00	
0.254	19.6	19.6
0.508	39.5	19.9
0.752	58.8	19.3
1.016	77.9	19.1
1.270	97.7	19.8
1.524	117.5	19.8
1.753	137.3	19.8
2.032	156.8	19.5
2.286	176.4	19.6
2.540	195.8	19.4

$f = 9193 \pm 5$ mc.
$\frac{1}{2}\lambda g_S = 2.327 \pm 0.002$-cm wavelength in the guide.
$\frac{1}{2}\lambda g_D = 2.340 \pm 0.005$-cm wavelength in the slotted line.
vswr for "E" and "H" arms ≈ 1.03
$2L_1$ = incremental path length of the through arms.
ϕ = relative phase shift in the slotted line.
$\Delta\phi$ = incremental phase shift in the slotted line.

is within 0.8°. The estimated maximum error in the plunger setting L_1 is ± 0.001 cm and in the location of the field minimum ± 0.005 cm. Thus the estimated maximum error in ϕ is $\pm 0.4°$. In view of this limited accuracy in the measurements it would seem that the phase shift was linear to better than 0.8°.

Conclusion

It has been shown that the symmetrical magic-tee makes a lossless linear phase changer whose phase linearity is affected to a second order for reflection coefficients equal to or less than 0.1.

The fact that isolation between the "E" and "H" plane arms is not infinite although usually greater than 40 db ($|S_{34}| < 0.01$) introduces additional errors. This coupling is caused by a certain amount of asymmetry for which S_{34} may be considered as a measure. Reference to (5)–(7) shows that when the tee is not perfectly symmetrical, $S_{34} \neq 0$ and the term $(S_{13}S_{14}S_{22} + S_{23}S_{24}S_{11} - S_{12}S_{13}S_{24} - S_{12}S_{14}S_{23})$ does not completely cancel. This term may however be expected to be small for slight asymmetry since it does cancel for a symmetrical tee. The main effect of small asymmetry is then to introduce two additional terms the order of magnitude of which is 0.02 and varying in phase with respect to the dominant term $S_{13}S_{14}(e^{-2j\theta_1} - e^{-2j\theta_2})$ in the numerator of (8). The dominant term has a magnitude of approximately unity, so the beating of these terms could be expected to produce an additional phase error of around $\pm \tan^{-1} 0.02$ or $\pm 1.14°$ if both perturbing terms add in phase. Thus a maximum phase error or departure from linearity of 2° may occur if all the errors add in phase and are 90° out of phase with the dominant wave. The experimental results show that in practice errors of less than 1° can be obtained using commercially available magic-tees.

Appendix

Since $[S]$ is unitary, it follows that

$$U_{11}^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 = 1 \tag{16}$$

$$2|S_{13}|^2 + U_{33}^2 = 1 \tag{17}$$

$$2|S_{14}|^2 + U_{44}^2 = 1 \tag{18}$$

$$U_{11}(S_{12}^* + S_{12}) - |S_{13}|^2 + |S_{14}|^2 = 0 \tag{19}$$

$$S_{13}^*(U_{11} - S_{12}) + S_{13}U_{33} = 0 \tag{20}$$

$$S_{14}^*(U_{11} + S_{12}) + S_{14}U_{44} = 0. \tag{21}$$

These six independent equations have nine unknown variables. It is then in general necessary to know the values of at least three of them to solve for all the others. Because the diagonal elements are the reflection coefficients measured in the four reference planes, they seem an appropriate choice of values to use.

From (17)

$$|S_{13}|^2 = \frac{1 - U_{33}^2}{2}. \tag{22}$$

From (18)

$$|S_{14}|^2 = \frac{1 - U_{44}^2}{2}. \tag{23}$$

From (19)

$$S_{12} + S_{12}^* = \frac{U_{44}^2 - U_{33}^2}{2U_{11}}.$$

Now

$$S_{12} = U_{12} + jV_{12},$$

$$S_{12} + S_{12}^* = 2U_{12},$$

and hence

$$U_{12} = \frac{U_{44}^2 - U_{33}^2}{4U_{11}}. \tag{24}$$

Substitution of (22) and (23) in (16) gives

$$|S_{12}|^2 = \frac{U_{33}^2 + U_{44}^2}{2} - U_{11}^2. \tag{25}$$

But

$$|S_{12}|^2 = U_{12}^2 + V_{12}^2 \tag{28}$$

whence

$$V_{12}^2 = \frac{U_{33}^2 + U_{44}^2}{2} - U_{11}^2 - U_{12}^2. \tag{27}$$

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